TEMPERATURE FIELD OF AN INFINITE PLATE IN THE CASE OF A VARIABLE HEAT-EXCHANGE COEFFICIENT

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An infinite system of ordinary differential equations is obtained; account of a finite number of these equations yields an approximate solution for the problem.

We are to solve the one-dimensional heat-conduction equation

$$\frac{\partial\theta(X,\ \text{Fo})}{\partial\,\text{Fo}} = \frac{\partial^2\theta\ (X,\text{Fo})}{\partial X^2} \tag{1}$$

with the initial conditions

$$\theta(X,\ 0) = f(X) \tag{2}$$

and the boundary conditions

$$\frac{\partial\theta(1,\text{Fo})}{\partial X} = \text{Bi}\ (\text{Fo})\ [\theta_m(\text{Fo}) - \theta(1,\text{Fo})], \tag{3}$$

$$\frac{\partial\theta(0,\ \text{Fo})}{\partial X} = 0. \tag{4}$$

We use the notation

$$q(\text{Fo}) = \text{Bi}(\text{Fo})[\theta_m(\text{Fo}) - \theta(1,\text{Fo})] \tag{5}$$

and rewrite condition (3) as

$$\frac{\partial\theta(1,\ \text{Fo})}{\partial X} = q(\text{Fo}). \tag{6}$$

Now solving Eq. (1) with boundary conditions (2) and (4) and boundary condition (6) of the second kind, we can find the relation between q(Fo) and θ(1, Fo), which, along with (5), gives the solution of problems (1)-(4).

The solution of problems (1), (2), (4), and (6) can be written [1]

$$\theta(X,\ \text{Fo}) = z_0(\text{Fo}) + \sum_{n=1}^{\infty}(-1)^n\cos n\pi X z_n(\text{Fo}), \tag{7}$$

where

$$z_0(\text{Fo}) = \int_0^1 f(X)\,dX + \int_0^{\text{Fo}} q(\tau)\,d\tau,$$

$$z_n(\text{Fo}) = \left[(-1)^n 2\int_0^1 f(X)\cos n\pi X dX + 2\int_0^{\text{Fo}} q(\tau)\exp(n\pi)^2\tau d\tau\right]\exp\left[-(n\pi)^2\text{Fo}\right], \tag{8}$$

$$n = 1, 2, \ldots$$

Differentiating the left and right sides of (8) with respect to Fo, we find a system of ordinary differential equations:

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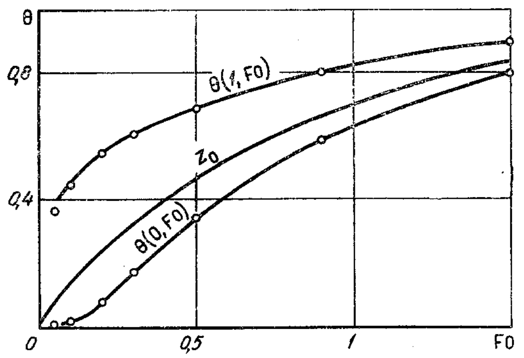

Fig. 1. Graph used to calculate the temperature of the plate surfaces.

$$\dot{z}_0(\mathrm{Fo}) = q(\mathrm{Fo}),$$
$$\left(\frac{1}{n\pi}\right)^2 \dot{z}_n(\mathrm{Fo}) + z_n(\mathrm{Fo}) = 2\left(\frac{1}{n\pi}\right)^2 q(\mathrm{Fo}),\ n = 1, 2, \ldots, \tag{9}$$

with the initial conditions

$$z_0(0) = \int_0^1 f(X)\,dX,$$
$$z_n(0) = (-1)^n 2\int_0^1 f(X)\cos n\pi X dX,\ n = 1, 2, \ldots \tag{10}$$

Substituting X = 1 into (7), and using $\cos n\pi = (-1)^n$ (n = 1, 2, ...), we find

$$\theta(1, \mathrm{Fo}) = z_0(\mathrm{Fo}) + \sum_{n=1}^{\infty} z_n(\mathrm{Fo}). \tag{11}$$

Now, we use (5) and (11) to convert system (9) to

$$\frac{1}{\mathrm{Bi}(\mathrm{Fo})}\dot{z}_0(\mathrm{Fo}) + z_0(\mathrm{Fo}) = \theta_m(\mathrm{Fo}) - \sum_{m=1}^{\infty} z_m(\mathrm{Fo}),$$

$$\left(\frac{1}{n\pi}\right)^2 \dot{z}_n(\mathrm{Fo}) + z_n(\mathrm{Fo}) = 2\,\mathrm{Bi}(\mathrm{Fo})\left(\frac{1}{n\pi}\right)^2\left[\theta_m(\mathrm{Fo}) - z_0(\mathrm{Fo}) - \sum_{m=1}^{\infty} z_m(\mathrm{Fo})\right],\ n = 1, 2, \ldots \tag{12}$$

Solving this system with the initial conditions (10), we can determine from Eq. (7) the temperature at any point X of the plate.

For approximate calculations, we can retain a finite number of the differential equations in system (12), because the time constants $(1/n\pi)^2$ (n = 1, 2, ...) decrease rapidly as n increases, and are negligible at large n ($1/\pi^2 = 0.1013$; $1/4\pi^2 = 0.0253$; $1/9\pi^2 = 0.0112$; $1/16\pi^2 = 0.063$; $1/25\pi^2 = 0.0040$; $1/36\pi^2 = 0.0028$). This truncated system of equations can be written

$$\frac{1}{\mathrm{Bi}(\mathrm{Fo})}\dot{z}_0(\mathrm{Fo}) + z_0(\mathrm{Fo}) = \theta_c(\mathrm{Fo}) - \sum_{m=1}^{\infty} z_m(\mathrm{Fo}),$$
$$\left(\frac{1}{n\pi}\right)^2 \dot{z}_n(\mathrm{Fo}) + z_n(\mathrm{Fo}) = 2\left(\frac{1}{n\pi}\right)^2 \dot{z}_0(\mathrm{Fo}),\ n = 1, 2, \ldots, s, \tag{13}$$
$$\sum_{n=s+1}^{\infty} z_n(\mathrm{Fo}) = 2\sum_{n=s+1}^{\infty}\left(\frac{1}{n\pi}\right)^2 \dot{z}_0(\mathrm{Fo}).$$

From the solution of this system, we find the temperature at point X of the plate, using the equation

$$\theta(X, \mathrm{Fo}) = z_0(\mathrm{Fo}) + \sum_{n=1}^{n=s}(-1)^n \cos n\pi X z_n(\mathrm{Fo}) + 2\sum_{n=s+1}^{\infty}(-1)^n\left(\frac{1}{n\pi}\right)^2 \cos n\pi X \dot{z}_0(\mathrm{Fo}). \tag{14}$$

However, since

$$2\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{n\pi}\right)^2\cos n\pi X = -\frac{1-3X^2}{6}, \tag{15}$$

Eq. (14) can be written

$$\theta(X, \mathrm{Fo}) = z_0(\mathrm{Fo}) + \sum_{n=1}^{n=s}(-1)^n \cos n\pi X z_n(\mathrm{Fo}) - \left[-\frac{1-3X^2}{6} + 2\sum_{n=1}^{n=s}(-1)^n\left(\frac{1}{n\pi}\right)^2\cos n\pi X\right]\dot{z}_0(\mathrm{Fo}). \tag{16}$$

Now substituting X = 1 into (15), we find

$$2\sum_{n=1}^{\infty}\left(\frac{1}{n\pi}\right)^2 = \frac{1}{3}.$$

Accordingly, we can write system (13) in the final form:

$$\begin{gathered} T(\mathrm{Fo})\dot{z}_0(\mathrm{Fo}) + z_0(\mathrm{Fo}) = \theta_m(\mathrm{Fo}) - \sum_{m=1}^{m=s} z_m(\mathrm{Fo}), \\ \left(\frac{1}{n\pi}\right)^2 \dot{z}_n(\mathrm{Fo}) + z_n(\mathrm{Fo}) = 2\left(\frac{1}{n\pi}\right)^2\dot{z}_0(\mathrm{Fo}),\ n = 1, 2, \ldots, s, \end{gathered} \tag{17}$$

where

$$T(\mathrm{Fo}) = \frac{1}{\mathrm{Bi}(\mathrm{Fo})} + \frac{1}{3} - 2\sum_{m=1}^{m=s}\left(\frac{1}{m\pi}\right)^2.$$

The approximate calculation of the temperature field of an infinite plate thus reduces to the solution of system (17) with the initial conditions (10), and the subsequent determination of the temperature at the given point from Eq. (16). In practice, it is sufficient to solve two or three equations (s = 1-2) of system (17).

Figure 1 shows the temperatures $\theta(1, \mathrm{Fo})$ and $\theta(0, \mathrm{Fo})$ of the plate surfaces determined for Bi = 2 with an account of only two equations (s = 1). The equations were integrated graphically by the Bashkirov method [2]. The points show temperature values obtained by an ordinary solution of systems (1)-(4). When Fo > 0.05, the results are in essentially complete agreement.

NOTATION

θ	is the plate temperature;
θ_m	is the medium temperature;
x	is the spatial coordinate;
t	is the time;
a	is the temperature conductivity;
λ	is the thermal conductivity;
$\alpha(t)$	is the heat-exchange coefficient;
L	is the plate thickness;
$X = x/L$	is the dimensionless coordinate;
$\mathrm{Fo} = at/L^2$	is the Fourier number;
$\mathrm{Bi(Fo)} = \alpha(\mathrm{Fo})L/\lambda$	is the Biot number.

LITERATURE CITED

1. A. V. Lykov, The Theory of Heat Conduction [in Russian], Izd. Vysshaya Shkola, Moscow (1967).
2. E. P. Popov, The Dynamics of Automatic-Control Systems [in Russian], Gostekhizdat, Moscow (1954).